



# The Ekeland variational principle for Henig proper minimizers and super minimizers<sup>☆</sup>

Truong Xuan Duc Ha

Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Viet Nam

## ARTICLE INFO

### Article history:

Received 15 December 2008

Available online 30 October 2009

Submitted by A. Daniilidis

Dedicated to my teacher, Professor

I.A. Bakhtin, on his seventy-sixth birthday

### Keywords:

Ekeland variational principle

Vector optimization

Henig proper minimizer

Super minimizer

Henig dilating cone

Cone with base

Set-valued map

Coderivative

## ABSTRACT

In this paper we consider, for the first time, approximate Henig proper minimizers and approximate super minimizers of a set-valued map  $F$  with values in a partially ordered vector space and formulate two versions of the Ekeland variational principle for these points involving coderivatives in the sense of Ioffe, Clarke and Mordukhovich. As applications we obtain sufficient conditions for  $F$  to have a Henig proper minimizer or a super minimizer under the Palais–Smale type conditions. The techniques are essentially based on the characterizations of Henig proper efficient points and super efficient points by mean of the Henig dilating cones and the Hiriart-Urruty signed distance function.

© 2009 Elsevier Inc. All rights reserved.

## 1. Introduction

The well-known Ekeland Variational Principle (EVP) [12] says roughly that for any lower semicontinuous (lsc) function  $f$  bounded from below on a complete metric space  $X$ , there exists an approximate minimizer of  $f$  which is an exact minimizer of a perturbed function. When  $X$  is a Banach space and  $f$  is Gâteaux differentiable, its derivative can be made arbitrarily small. Moreover, if  $f$  satisfies the Palais–Smale condition then it attains a minimum on  $X$ .

During the last three decades many authors have been interested in extending EVP and related results to a single- or set-valued map with values in a vector space partially ordered by a convex cone, see [4,5,7–9,11,13,14,16–21,23–25,28,29,32,33, 35,38,46,47,49,53] and the references therein. A brief enquiry into the matter reveals that the authors concentrated mainly on exact/approximate Pareto minimizers or exact/approximate weak minimizers, and the ordering cone was often assumed to have a nonempty interior. However, this assumption does not hold in many cases and several authors tried to drop it, see, for instance, [4,5,14,19]. In [4,5] Bao and Mordukhovich obtained subdifferential versions of EVP for approximate minimizers of a set-valued map  $F$  in Asplund space settings as well as sufficient conditions for  $F$  to have intrinsic relative minimizers or primary relative Pareto minimizers under a refined subdifferential Palais–Smale condition. In [19] we established EVP for exact/approximate positive proper minimizers of a set-valued map  $F$  taking values in a Banach space with the ordering cone admitting strictly positive functionals.

<sup>☆</sup> The author is grateful to Professor B. Mordukhovich for comments on the first version of the manuscript that helped her to improve the original presentation. The author thanks the referees for valuable remarks and comments. The author thanks Professor V.V. Gorokhovich for comments and the reference Gorokhovich (1990) [15].

E-mail address: txdha@math.ac.vn.

In this paper, we consider, for the first time, approximate Henig proper minimizers and approximate super minimizers of a set-valued map  $F$ . We formulate two theorems about EVP for these points involving the Ioffe approximate coderivative, the Clarke coderivative and the Mordukhovich coderivative of  $F$ , and apply them to obtain sufficient conditions for  $F$  to have a Henig proper minimizer or a super minimizer under the Palais–Smale type conditions. Specifically, we work in a broad class of Banach spaces and Asplund spaces in which the ordering cones have a base and are normal and, when it concerns with super efficiency, the ordering cones have a bounded base. The techniques are essentially based on the characterizations of Henig proper efficient points and super efficient points by mean of the Henig dilating cones and the Hiriart-Urruty signed distance function.

The rest of the paper is organized as follows. In Section 2 we recall the notions of normal cone, subdifferential and coderivative in the sense of Ioffe, Clarke and Mordukhovich. Section 3 is devoted to the characterizations of Henig proper efficient points and super efficient points. In Section 4 we present the concepts of approximate Henig proper minimizers and approximate super minimizers of a set-valued map  $F$  and formulate EVP for these points. In Section 5 we establish sufficient conditions for  $F$  to have a Henig proper minimizer or a super minimizer under the Palais–Smale type conditions.

## 2. Normal cone, subdifferential and coderivative in the senses of Ioffe, Clarke and Mordukhovich

For the convenience of the reader we repeat the relevant materials from [10,30,31,37,41–45,48].

Throughout the paper  $X$  and  $Y$  are Banach spaces with their norms  $\|\cdot\|$  and duals  $X^*$  and  $Y^*$ , respectively. The closed unit ball and the open unit ball in any space, say  $X$ , are denoted by  $B_X$  and  $\tilde{B}_X$ , respectively; when no confusion occurs, they are simply denoted by  $B$  and  $\tilde{B}$ , respectively. For a nonempty set  $A$  in any space, say in  $X$ , we will use the following notations:  $\text{int } A$  and  $\text{cl } A$  stand for the interior and closure of  $A$ ,

$$\text{cone } A = \{ta : t \in \mathbb{R}_+, a \in A\},$$

where  $\mathbb{R}_+ = [0, \infty[$ ,  $d_A$  is the distance function  $d_A(x) = \inf\{\|x - a\| : a \in A\}$  and  $\chi_A$  is the indicator function

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that a Banach space is *Asplund* if each of its separable subspace has a separable dual. This class has been comprehensively investigated in geometric theory of Banach spaces and has been largely employed in variational analysis; see, e.g. [44,45].

Given a function  $g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , its domain and epigraph are the sets  $\text{dom } g = \{x \in X : g \text{ is finite at } x\}$  and  $\text{epi } g = \{(x, t) \in X \times \mathbb{R} : g(x) \leq t\}$ , respectively.

We recall the concepts of the Ioffe approximate subdifferential and approximate normal cone [30,31]. The approximate subdifferential is first defined for Lipschitz functions. Namely, supposing that  $g$  is locally Lipschitz near  $x \in \text{dom } g$ , the *Ioffe approximate subdifferential* of  $g$  at  $x$  [30] is the set

$$\partial_A g(x) = \bigcap_{L \in \mathcal{F}} \limsup_{(\epsilon, y) \rightarrow (0^+, x)} \partial_{\epsilon}^- g_{y+L}(y),$$

where  $\mathcal{F}$  is the collection of all finite-dimensional subspaces of  $X$ ,  $g_{y+L}(u) = g(u)$  if  $u \in y+L$  and  $g_{y+L}(u) = +\infty$  otherwise, for  $\epsilon \geq 0$

$$\partial_{\epsilon}^- g_{y+L}(y) = \left\{ x^* \in X^* : x^*(v) \leq \epsilon \|v\| + \liminf_{t \rightarrow 0^+} t^{-1} [g_{y+L}(y + tv) - g_{y+L}(y)], \forall v \in X \right\}.$$

Let  $\Omega$  be a nonempty subset of  $X$  different from  $X$  and  $x \in \text{cl } \Omega$ . The *Ioffe approximate normal cone* to  $\Omega$  at  $x \in \Omega$  [30] is given by

$$N_A(x; \Omega) = \bigcup_{\lambda > 0} \lambda \partial_A d(x; \Omega).$$

Now approximate subdifferential can be defined for an arbitrary lsc function through the normal cone to its epigraph as follows

$$\partial_A g(x) = \{x^* \in X^* : (x^*, -1) \in N_A((x, g(x)); \text{epi } g)\}.$$

Next, we recall the concepts of the Clarke subdifferential and normal cone [10]. The Clarke subdifferential is first defined for Lipschitz functions. Namely, supposing that  $g$  is locally Lipschitz near  $x$ , the *Clarke generalized subdifferential* of  $g$  at  $x$  is the set

$$\partial_C g(x) = \{x^* \in X^* : x^*(v) \leq g^0(x; v), \forall v \in X\},$$

where  $g^0(x; v)$  is the generalized directional derivative of  $g$  at  $x$  in the direction  $v$

$$g^0(x; v) = \limsup_{y \rightarrow x, t \rightarrow 0^+} \frac{g(y + tv) - g(y)}{t}.$$

The Clarke normal cone to  $S$  at  $x \in \Omega$  is given by

$$N_C(x; \Omega) = \text{cl} \bigcup_{\lambda > 0} \lambda \partial_C d(x; \Omega).$$

Now the Clarke subdifferential can be defined for an arbitrary lsc function through the normal cone to its epigraph as follows

$$\partial_C g(x) = \{x^* \in X^*: (x^*, -1) \in N_C((x, g(x)); \text{epi } g)\}.$$

Further we recall the concepts of the Mordukhovich subdifferential and normal cone [37,41–45]. The set of Fréchet  $\epsilon$ -normals to  $\Omega$  at  $x$  is given by

$$\hat{N}_\epsilon(x; \Omega) = \left\{ x^* \in X^*: \limsup_{x' \xrightarrow{\Omega} x} \frac{x^*(x' - x)}{\|x' - x\|} \leq \epsilon \right\}.$$

When  $\epsilon = 0$ , this set is a cone which is called the Fréchet normal cone to  $\Omega$  at  $x$  and is denoted by  $\hat{N}(x; \Omega)$ . The set of limiting Fréchet  $\epsilon$ -normals to  $\Omega$  at  $x$  is given by

$$N_\epsilon(x; \Omega) = \limsup_{x' \xrightarrow{\Omega} x} \hat{N}_\epsilon(x'; \Omega),$$

where the limit in the right-hand side means the sequential Kuratowski–Painlevé upper limit with respect to the norm topology in  $X$  and the weak-star  $\omega^*$  topology in  $X^*$ . The Mordukhovich normal cone to  $\Omega$  at  $x$  is defined by

$$N_M(x; \Omega) = \limsup_{x' \xrightarrow{\Omega} x, \epsilon \downarrow 0^+} \hat{N}_\epsilon(x'; \Omega).$$

Supposing that  $g$  is lsc, the Mordukhovich subdifferential of  $g$  at  $x$  is the set

$$\partial_M g(x) = \{x^* \in X^*: (x^*, -1) \in N_M((x, g(x)); \text{epi } g)\}.$$

Note that when  $g$  is convex and Lipschitz near  $x$ , the above subdifferentials reduce to the subdifferential of convex analysis  $\partial g$ , i.e.,

$$\partial g(x) = \{x^* \in X^*: x^*(x') - x^*(x) \leq g(x') - g(x) \text{ for all } x' \in \text{dom } g\}.$$

Throughout the paper  $F: X \rightrightarrows Y$  is a map with set values. For the sake of convenience we assume that  $F(x)$  is nonempty for all  $x \in X$ . We denote its graph by  $\text{gr } F$ , i.e.,  $\text{gr } F = \{(x, y) \in X \times Y: y \in F(x)\}$ . Recall that  $F$  is upper semicontinuous (usc) at  $x \in X$  if for any open set  $V$  with  $F(x) \subset V$  there exists an open set  $U$  with  $x \in U$  such that  $F(x') \subset V$  for all  $x' \in U$  and that  $F$  is usc on  $X$  if it is usc at every  $x \in X$ .

Let us recall the concept of coderivatives for the map  $F$  generated by the above normal cones to its graph [30,31,37,41–45]. Suppose that  $F$  has a closed graph. The Ioffe approximate coderivative, the Clarke coderivative and the Mordukhovich coderivative of  $F$  at  $(x, y) \in \text{gr } F$  are the set-valued maps  $D_A^* F(x, y)$ ,  $D_C^* F(x, y)$  and  $D_M^* F(x, y)$  from  $Y^*$  into  $X^*$  defined by

$$\begin{aligned} D_A^* F(x, y)(y^*) &= \{x^* \in X^*: (x^*, -y^*) \in N_A((x, y); \text{gr } F)\}, \\ D_C^* F(x, y)(y^*) &= \{x^* \in X^*: (x^*, -y^*) \in N_C((x, y); \text{gr } F)\} \end{aligned}$$

and

$$D_M^* F(x, y)(y^*) = \{x^* \in X^*: (x^*, -y^*) \in N_M((x, y); \text{gr } F)\},$$

respectively.

**Remark 2.1.** Note that in [42] Mordukhovich introduced the notion of coderivatives of a set-valued map regardless of the normal cone used. After he suggested this approach to differentiability of maps, we may consider different specific coderivatives generated by different normal cones. The Mordukhovich coderivative related to a normal cone in a finite-dimensional space was introduced in [41]. This cone was extended to Banach spaces in [37]. We mention that Clarke neither introduced nor used any coderivative concept for either set-valued or single-valued maps, but the coderivative generated by the Clarke normal cone in the scheme of [42] as above has been used under the name “Clarke’s coderivative” in [43].

In the sequel, for the sake of convenience, we made *the convention* that the same notations  $\partial g$  and  $D^*F$  are used for the above kinds of subdifferentials and coderivatives and that the spaces under consideration are Asplund whenever the subdifferential and the coderivative are understood in the sense of Mordukhovich.

Among all the properties of the above subdifferentials let us recall those which will be used in the sequel [10,30,31,48].

### Proposition 2.1.

- (i)  $\partial \chi_\Omega(x) = N(x; \Omega)$ .
- (ii) If  $g(x') \geq g(x)$  for all  $x'$  in a neighborhood of  $x \in \text{dom } g$ , then  $0 \in \partial g(x)$ .
- (iii) (The sum rule.) If  $h: X \rightarrow R \cup \{+\infty\}$  is Lipschitz near  $x \in \text{dom } g \cap \text{dom } h$  then  $\partial(g+h)(x) \subset \partial g(x) + \partial h(x)$ .
- (iv) For the norm  $\|\cdot\|$  in  $X$  one has  $\partial\|\cdot\|(0) = B_{X^*}$ .

### 3. The Henig proper efficiency and the super efficiency

This section is devoted to Henig proper efficient points and super efficient points of a set. We characterize these points by mean of the Henig dilating cones and the Hiriart-Urruty signed distance function. We also establish some useful properties of these cones and function.

Throughout the paper  $K \subset Y$  is a convex, pointed and closed cone (pointedness means  $K \cap (-K) = \{0\}$ ). This cone induces a partial order on  $Y$ : For  $y_1, y_2 \in Y$  we write

$$y_1 \leq_K y_2 \quad \text{if } y_2 - y_1 \in K.$$

Recall [3,36] that  $K$  is *normal* if there exists a scalar  $N > 0$  such that for any  $k_1, k_2 \in K$  with  $k_1 \leq_K k_2$  one has  $\|k_1\| \leq N\|k_2\|$ .

We say that a convex set  $\Theta \subset Y$  is a *base* of  $K$  if  $0 \notin \text{cl } \Theta$  and

$$K = \{t\theta: t \in R_+, \theta \in \Theta\}.$$

When  $\Theta$  is bounded, we say that  $K$  has a *bounded base*.

Denote

$$K^{+i} = \{\varphi \in Y^*: \varphi(k) > 0, \forall k \in K \setminus \{0\}\}.$$

It is known that  $K$  has a base iff  $K^{+i} \neq \emptyset$  and if  $K$  has a bounded base then  $K$  is normal [36].

We provide examples of ordering cones in some classical Banach spaces.

**Example 3.1.** Consider the following Banach spaces:  $R^n$  – the  $n$ -dimensional euclidean space,  $C_{[0,1]}$  – the space of continuous functions on  $[0, 1]$ ,  $L^p_{[0,1]}$  ( $1 \leq p < \infty$ ) – the space of  $p$ -integrable functions on  $[0, 1]$ ,  $l^p$  ( $1 \leq p < \infty$ ) – the space of  $p$ -summable sequences,  $m$  – the space of bounded sequences,  $c$  – the space of convergent sequences and  $c_0$  – the space of null sequences (i.e., sequences which converge to zero). Let  $K$  be the nonnegative orthant in the corresponding space. It is known that the nonnegative orthants in  $R^n$ ,  $C_{[0,1]}$ ,  $L^p_{[0,1]}$ ,  $l^p$  ( $1 \leq p < \infty$ ),  $m$ ,  $c$  and  $c_0$  have a base and are normal, the nonnegative orthants in  $R^n$ ,  $L^1_{[0,1]}$  and  $l^1$  have a bounded base and only in  $R^n$  and  $C_{[0,1]}$  the nonnegative orthants have a nonempty interior [3,36].

Let  $\Theta$  be the base of  $K$  as before. Set

$$\delta = \inf\{\|\theta\|: \theta \in \Theta\} > 0.$$

For each scalar  $\eta \in ]0, \delta[$  we associate to  $K$  a cone

$$K_\eta = \text{cl cone}(\Theta + \eta B).$$

This cone, known as the Henig dilating cone [26], plays an important role in the study of proper efficiency and will be frequently used in our work. Note that  $K_\eta$  is convex, pointed and closed. By Theorem 1.1 in [6],  $\text{cl}(\Theta + \eta B)$  is a base of  $K_\eta$ , i.e.,

$$K_\eta = \text{cone cl}(\Theta + \eta B). \quad (1)$$

Throughout this section let  $A$  be a nonempty subset of  $Y$ , different from  $Y$ . In this paper we consider the following concepts of efficiency [6,26,34,39,51,52,54].

**Definition 3.1.** Let  $\bar{a} \in A$ . We say that

- (i)  $\bar{a}$  is an efficient (or Pareto minimal) point of  $A$  with respect to (wrt)  $K$  ( $\bar{a} \in \text{Min}(A; K)$ ) if  $(A - \bar{a}) \cap (-K \setminus \{0\}) = \emptyset$ ;
- (ii) supposing that  $\text{int } K \neq \emptyset$ ,  $\bar{a}$  is a weak efficient point of  $A$  wrt  $K$  ( $\bar{a} \in \text{WMin}(A; K)$ ) if  $(A - \bar{a}) \cap (-\text{int } K) = \emptyset$ ;
- (iii) supposing that  $K$  has a base  $\Theta$ ,  $\bar{a}$  is a Henig proper efficient point of  $A$  wrt  $\Theta$  ( $\bar{a} \in \text{He}(A; \Theta)$ ) if there exists a scalar  $\eta \in ]0, \delta[$  such that

$$\text{cl cone}(A - \bar{a}) \cap (-K_\eta) = \{0\};$$

- (iv)  $\bar{a}$  is a super efficient point of  $A$  wrt  $K$  ( $\bar{a} \in \text{SE}(A; K)$ ) if there is a scalar  $\rho > 0$  such that

$$\text{cl cone}(A - \bar{a}) \cap (B - K) \subseteq \rho B.$$

In the sequel, we *always assume that  $K$  has a base  $\Theta$* . When no confusion occurs, we simply write  $\text{Min}(A)$ ,  $\text{WMin}(A)$ ,  $\text{He}(A)$  and  $\text{SE}(A)$  for the sets of efficient points in Definition 3.1. Moreover, when speaking of weak efficient points we mean that  $\text{int } K$  is nonempty and when saying that  $K$  has a bounded base we mean that  $\Theta$  is bounded. The above definition of Henig proper efficient points expressed in terms of the base  $\Theta$  can be found in [6], see also [51,52,54]. Note that in [52] Zheng used the notation  $\text{He}(A; K)$  instead of  $\text{He}(A; \Theta)$ . For an equivalent definition of Henig proper efficient points by means of a functional from  $K^{+i}$  the reader is referred to [40].

From the definition, one has

$$\text{He}(A) \subseteq \text{Min}(A) \subseteq \text{WMin}(A).$$

The following result is a key tool for obtaining geometric characterizations of Henig proper efficient points and super efficient points.

**Proposition 3.1.**

- (i)  $A \cap (-K_\eta) \subseteq \{0\}$  for some  $\eta \in ]0, \delta[$  implies  $\text{cl cone } A \cap (-K_{\eta'}) = \{0\}$  for any  $\eta' \in ]0, \eta[$ .
- (ii)  $\text{cl cone } A \cap (B - K) \subseteq \rho B$  for some  $\rho > 0$  implies  $\text{cl cone } A \cap (-K_\eta) = \{0\}$  for any  $\eta \in ]0, \delta/(\rho + 1)[$ .
- (iii) Supposing that  $\Theta$  is bounded,  $\text{cl cone } A \cap (-K_\eta) = \{0\}$  for some  $\eta \in ]0, \delta[$  implies  $\text{cl cone } A \cap (B - K) \subseteq \rho B$ , where  $\rho = 1 + \bar{\delta}/\eta$  and  $\bar{\delta} = \sup\{\|\theta\|; \theta \in \Theta\}$ .

**Proof.** (i) Arguing by contradiction, suppose that  $A \cap (-K_\eta) \subseteq \{0\}$  for some  $\eta \in ]0, \delta[$  but there exists  $\eta' \in ]0, \eta[$  such that the relation

$$\text{cl cone } A \cap (-K_{\eta'}) = \{0\} \tag{2}$$

is not true. Then there exists  $z \in [\text{cl cone } A \cap (-K_{\eta'})] \setminus \{0\}$ . As  $z \in -K_{\eta'} \setminus \{0\}$ , (1) applied to  $K_{\eta'}$  yields the existence of  $\lambda > 0$  and  $\omega \in \text{cl}(\Theta + \eta'B)$  such that  $z = -\lambda\omega$ . Therefore, one can find  $u \in Y$  with  $\|u\| \leq (\eta - \eta')/2$ ,  $\theta \in \Theta$  and  $b' \in B$  such that

$$z = -\lambda(\theta + \eta'b' + u). \tag{3}$$

On the other hand, as  $z \in \text{cl cone } A$ , there exist  $\gamma \geq 0$ ,  $a \in A$  and  $v \in Y$  such that  $\|v\| \leq \min\{\lambda(\eta - \eta')/2, \|z\|/2\}$  and  $z = \gamma a + v$ . Thus, we have

$$z = -\lambda(\theta + \eta'b' + u) = \gamma a + v. \tag{4}$$

Observing that  $\|\eta'b' + u + v/\lambda\| \leq \eta' + (\eta - \eta')/2 + (\eta - \eta')/2 = \eta$ , we can find  $b \in B$  such that  $\eta b = \eta'b' + u + v/\lambda$ . Therefore, (4) gives

$$\gamma a = -\lambda(\theta + \eta'b' + u + v/\lambda) = -\lambda(\theta + \eta b).$$

Since  $\gamma\|a\| = \|z - v\| \geq \|z\| - \|v\| \geq \|z\|/2 > 0$ , we have  $\gamma > 0$  and  $a \neq 0$ . It follows then that  $a \in [A \cap (-K_\eta)] \setminus \{0\}$ . This contradicts  $A \cap (-K_\eta) \subseteq \{0\}$  and thus justifies (2).

(ii) Arguing by contradiction, suppose that  $\text{cl cone } A \cap (B - K) \subseteq \rho B$  for some  $\rho > 0$  but (2) does not hold for some  $\eta' \in ]0, \delta/(\rho + 1)[$ . Then there exists  $z \in [\text{cl cone } A \cap (-K_{\eta'})] \setminus \{0\}$ . Choose an arbitrary  $\eta \in ]\eta', \delta/(\rho + 1)[$ . Let  $\lambda > 0$ ,  $\theta \in \Theta$ ,  $u \in Y$  with  $\|u\| \leq \eta - \eta'$  and  $b' \in B$  such that (3) holds. Since  $\|\eta'b' + u\| \leq \eta' + \eta - \eta' = \eta$ , there exists  $b \in B$  such that  $\eta b = \eta'b' + u$  and  $z = -\lambda(\theta + \eta b)$ . Therefore, we have

$$\|z\| = \lambda\|\theta + \eta b\| \geq \lambda(\|\theta\| - \|\eta b\|) \geq \lambda(\delta - \eta).$$

Denote  $\bar{z} = z/(\lambda\eta)$ . Since  $\delta/\eta > \rho + 1$ , we obtain

$$\|\bar{z}\| = \|z\|/(\lambda\eta) \geq (\delta - \eta)/\eta = \delta/\eta - 1 > \rho + 1 - 1 = \rho.$$

Hence,  $\|\bar{z}\| > \rho$ . On the other hand, it is easy to see that  $\bar{z} \in \text{cl cone } A$  and  $\bar{z} = -\theta/\eta - b \in B - K$  and, therefore,  $\bar{z} \in \text{cl cone } A \cap (B - K)$ . Since  $\text{cl cone } A \cap (B - K) \subseteq \rho B$ , we arrive at  $\|\bar{z}\| \leq \rho$ , a contradiction.

(iii) Suppose that  $\text{cl cone } A \cap (-K_\eta) = \{0\}$  for some  $\eta \in ]0, \delta[$ . Let  $z \in \text{cl cone } A \cap (B - K)$  be an arbitrary nonzero vector. Let  $b \in B$ ,  $\lambda > 0$ , and  $\theta \in \Theta$  such that  $z = -\lambda\theta + b$ . Then  $1/\lambda = \|\theta\|/\|z - b\| \leq \bar{\delta}/(\|z\| - 1)$ . If  $\|z\| > 1 + \bar{\delta}/\eta$  then we get  $1/\lambda \leq \bar{\delta}/(\|z\| - 1) < \eta$ ,  $b/\lambda \in \eta B$  and  $z = \lambda(-\theta + b/\lambda) \in -K_\eta$ . This contradicts  $\text{cl cone } A \cap (-K_\eta) = \{0\}$  and thus implies  $\|z\| \leq 1 + \bar{\delta}/\eta$ .  $\square$

Let us state some consequences of Proposition 3.1 that will be used later. Firstly, we deduce the following known useful relations.

**Proposition 3.2.**

$$\text{SE}(A) \subseteq \text{He}(A)$$

and

$$\text{SE}(A) = \text{He}(A) \quad \text{if } \Theta \text{ is bounded.}$$

**Proof.** Let  $\bar{a} \in \text{SE}(A)$ . By the definition, there is a scalar  $\rho > 0$  such that  $\text{cl cone}(A - \bar{a}) \cap (B - K) \subseteq \rho B$ . Applying Proposition 3.1(ii) to the set  $A - \bar{a}$  in the place of  $A$  we get  $\text{cl cone}(A - \bar{a}) \cap (-K_\eta) = \{0\}$  for any  $\eta \in ]0, \delta/(\rho + 1)[$ . Therefore,  $\bar{a} \in \text{He}(A)$ .

Next, suppose that  $\Theta$  is bounded and  $\bar{a} \in \text{He}(A)$ . By the definition, there exists a scalar  $\eta \in ]0, \delta[$  such that  $\text{cl cone}(A - \bar{a}) \cap (-K_\eta) = \{0\}$ . Applying Proposition 3.1(iii) to the set  $A - \bar{a}$  in the place of  $A$  we get  $\text{cl cone}(A - \bar{a}) \cap (B - K) \subseteq \rho B_Y$  for  $\rho = 1 + \bar{\delta}/\eta$ , where  $\bar{\delta} = \sup\{\|\theta\|; \theta \in \Theta\}$ . Therefore,  $\bar{a} \in \text{SE}(A)$ .  $\square$

Proposition 3.1 allows us to equivalently describe a Henig proper efficient point and a super efficient point as an efficient point not wrt  $K$  but wrt some Henig dilating cone. This fact plays an important role in our study.

**Proposition 3.3.**

- (i)  $\bar{a} \in \text{He}(A)$  iff  $\bar{a} \in \text{Min}(A; K_\eta)$  for some  $\eta \in ]0, \delta[$ .
- (ii) Supposing that  $\Theta$  is bounded,  $\bar{a} \in \text{SE}(A)$  iff  $\bar{a} \in \text{Min}(A; K_\eta)$  for some  $\eta \in ]0, \delta[$ .

**Proof.** (i) If  $\bar{a} \in \text{He}(A)$  then  $\text{cl cone}(A - \bar{a}) \cap (-K_\eta) = \{0\}$  for some  $\eta \in ]0, \delta[$  and hence,  $(A - \bar{a}) \cap (-K_\eta) = \{0\}$ , which means that  $\bar{a} \in \text{Min}(A; K_\eta)$ . Conversely, if  $\bar{a} \in \text{Min}(A; K_\eta)$  for some  $\eta \in ]0, \delta[$  then  $(A - \bar{a}) \cap (-K_\eta) = \{0\}$ . This and Proposition 3.1(i) applied to the set  $A - \bar{a}$  in the place of  $A$  yield that  $\text{cl cone}(A - \bar{a}) \cap (-K_{\eta'}) = \{0\}$  for any  $\eta' \in ]0, \eta[$ . Thus,  $\bar{a} \in \text{He}(A)$ .

(ii) The assertion follows immediately from the assertion (i) and Proposition 3.2.  $\square$

Our next aim is to characterize a Henig proper efficient point and a super efficient point by the so-called signed distance function. Given a nonempty set  $A$  in  $Y$ , Hiriart-Urruty [27] defined this function, denoted by  $\Delta_A$ , as follows: for  $y \in Y$ ,

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y).$$

We list some properties of the function  $\Delta_A$  in the following proposition (see [27,50]). Some nice and useful properties of the subdifferential of this function in several special cases will be established in the end of the section.

**Proposition 3.4.** Assume that  $A$  is closed. Then

- (i) The function  $\Delta_A$  is 1-Lipschitz.
- (ii) The function  $\Delta_A$  is convex when  $A$  is convex and it is positively homogeneous when  $A$  is a cone.
- (iii) In particular, when  $A = -K$  and  $K$  is a closed convex cone, the function  $\Delta_{-K}$  is monotone, i.e.,

$$\Delta_{-K}(y_1) \leq \Delta_{-K}(y_2) \quad \text{for all } y_1, y_2 \in Y \text{ such that } y_1 \leq y_2$$

and satisfies the triangle inequality, i.e.,

$$\Delta_{-K}(y_1 + y_2) \leq \Delta_{-K}(y_1) + \Delta_{-K}(y_2) \quad \text{for all } y_1, y_2 \in Y.$$

**Proof.** The first two assertions are established in [27] and the last one can be easily derived from the formers.  $\square$

Gorokhovich [15] and Zaffaroni [50] have obtained the characterization of an efficient point and a weak efficient point by mean of the function  $\Delta_{-K}$ . Below we use Proposition 3.3 and the function  $\Delta_{-K_\eta}$  to characterize a Henig proper efficient

point and a super efficient point. We refer the interested readers to our recent work [22], where the signed distance function associated to different open cones was used to characterize not only the above efficient points but also efficient points in the sense of Benson, Borwein, Hurwicz and Hartley.

**Proposition 3.5.**

(i)  $\bar{a} \in \text{He}(A)$  iff there exists  $\eta \in ]0, \delta[$  such that

$$\Delta_{-K_\eta}(a - \bar{a}) \geq \Delta_{-K_\eta}(0) = 0 \quad \text{for all } a \in A. \quad (5)$$

(ii) Supposing that  $\Theta$  is bounded,  $\bar{a} \in \text{SE}(A)$  if there exists  $\eta \in ]0, \delta[$  such that (5) holds.

**Proof.** Observe first that since  $K_\eta$  is closed and 0 belongs to the boundary of  $K_\eta$ , we get  $\Delta_{-K_\eta}(0) = 0$ .

(i) Suppose that  $\bar{a} \in \text{He}(A)$ . Proposition 3.3(i) implies that there is some  $\eta \in ]0, \delta[$  such that  $\bar{a} \in \text{Min}(A; K_\eta)$ . By the definition,  $(A - \bar{a}) \cap (-K_\eta) = \{0\}$ . Then for any  $a \in A$ ,  $a \neq \bar{a}$  we have  $a - \bar{a} \notin -K_\eta$  and hence,

$$\Delta_{-K_\eta}(a - \bar{a}) = d_{-K_\eta}(a - \bar{a}) - d_{Y \setminus (-K_\eta)}(a - \bar{a}) = d_{-K_\eta}(a - \bar{a}) \geq 0.$$

Thus, (5) holds. Next, suppose that (5) holds for some  $\eta \in ]0, \delta[$ . For  $\eta' \in ]0, \eta[$ , we claim that

$$(A - \bar{a}) \cap (-K_{\eta'}) = \{0\}. \quad (6)$$

Indeed, if (6) does not hold then there exists  $a \in A$ ,  $a \neq \bar{a}$  such that  $a - \bar{a} \in -K_{\eta'}$ . By (1), there exist  $\lambda > 0$ ,  $u \in Y$  with  $\|u\| \leq (\eta - \eta')/4$ ,  $\theta \in \Theta$  and  $b' \in B$  such that  $a - \bar{a} = -\lambda(\theta + \eta'b' + u)$ . Since  $\|\eta'b' + u\| \leq \eta' + (\eta - \eta')/4$ , it is easy to see that  $\eta'b' + u + (\eta - \eta')/4B \subset (\eta + \eta')/2B$  and that  $\theta + \eta'b' + u + (\eta - \eta')/4B \subset \Theta + \eta\tilde{B}$ . Therefore,  $a - \bar{a} \in \text{int}(-K_\eta)$ . It follows that  $d_{Y \setminus (-K_\eta)}(a - \bar{a}) > 0$ , which yields

$$\Delta_{-K_\eta}(a - \bar{a}) = d_{-K_\eta}(a - \bar{a}) - d_{Y \setminus (-K_\eta)}(a - \bar{a}) = -d_{Y \setminus (-K_\eta)}(a - \bar{a}) < 0.$$

This contradicts (5) and thus justifies (6). Now, (6) and Proposition 3.3(i) yield that  $\bar{a} \in \text{He}(A)$ .

(ii) Suppose that (5) holds for some  $\eta \in ]0, \delta[$ . Then (i) yields that  $\bar{a} \in \text{He}(A)$ . As  $\Theta$  is bounded, Proposition 3.2 implies that  $\bar{a} \in \text{SE}(A)$ .  $\square$

We conclude this section by establishing some properties of the subdifferential of the function  $\Delta_{-K_\eta}$  which will play an important role in formulating our EVP.

**Proposition 3.6.** Let  $\eta \in ]0, \delta[$ . Then

(i)  $\partial\Delta_{-K_\eta}(0) \subset K^{+i}$  and  $\eta/\delta \leq \|y^*\| \leq 1$  for all  $y^* \in \partial\Delta_{-K_\eta}(0)$ .

(ii) Supposing that  $\Theta$  is bounded, we have  $\partial\Delta_{-K_\eta}(0) \subset \text{int } K^+$ .

**Proof.** (i) Note that the inclusion  $\partial\Delta_{-K_\eta}(0) \subset K^{+i}$  has already been established with a slightly modified argument in [22]. Our aim is to show that  $y^* \in \partial\Delta_{-K_\eta}(0)$  implies  $y^* \in K^{+i}$ . Recall that for any  $\theta \in \Theta$ , we have  $-(\theta + \eta\tilde{B}) \subset -K_\eta$ . Hence,  $d_{Y \setminus (-K_\eta)}(-\theta) \geq \eta$ . The definition of the convex subdifferential yields

$$y^*(-\theta) \leq \Delta_{-K_\eta}(-\theta) - \Delta_{-K_\eta}(0) = d_{-K_\eta}(-\theta) - d_{Y \setminus (-K_\eta)}(-\theta) = -d_{Y \setminus (-K_\eta)}(-\theta) \leq -\eta.$$

It follows then that

$$y^*(\theta) \geq \eta \quad \text{for all } \theta \in \Theta. \quad (7)$$

Now, let  $k \in K \setminus \{0\}$ . As  $\Theta$  is a base of  $K$ , there exist a scalar  $t > 0$  and  $\theta \in \Theta$  such that  $k = t\theta$ . Then (7) yields  $y^*(k) = y^*(t\theta) = ty^*(\theta) > 0$ . Therefore,  $y^* \in K^{+i}$ .

Further, as the function  $\Delta_{-K_\eta}$  is 1-Lipschitz, we get  $\|y^*\| \leq 1$ . From (7) we deduce that

$$\|y^*\| = \sup_{\|y\|=1} y^*(y) \geq y^*(\theta/\|\theta\|) \geq \eta/\|\theta\| \quad \text{for all } \theta \in \Theta,$$

which implies  $\|y^*\| \geq \eta/\delta$ , as it was to be shown.

(ii) Let  $y^* \in \partial\Delta_{-K_\eta}(0)$ . We claim that  $y^*(k) \geq \eta/\delta\|k\|$  for all  $k \in K \setminus \{0\}$ , which means that  $y^*$  is uniformly positive on  $K$  (see [3,36]). Indeed, if  $k \in K \setminus \{0\}$  then  $k = t\theta$  for some scalar  $t > 0$  and  $\theta \in \Theta$ . One has  $t = \|k\|/\|\theta\| \geq \|k\|/\delta$ , which together with (7) imply

$$y^*(k) = ty^*(\theta) \geq (\eta/\delta)\|k\|,$$

as it was to be shown. Hence,  $y^* \in \text{int } K^+$  [36].  $\square$

#### 4. Variants of EVP for approximate Henig proper minimizers and approximate super minimizers

Let  $F$  be, as before, a set-valued map with nonempty values from  $X$  to  $Y$ . In this section we formulate two variants of EVP for approximate Henig proper minimizers and approximate super minimizers of the map  $F$  which involve the Clarke normal cone to the graph of  $F$ .

Let us recall some concepts of exact/approximate minimizers of a set-valued map. Note that the concepts of efficient points of a set in Definition 3.1 naturally induce the following concepts of minimizers of  $F$ . Put  $F(X) = \bigcup_{x \in X} F(x)$ .

**Definition 4.1.** We say that  $(\bar{x}, \bar{y}) \in \text{gr } F$  is

- (i) a minimizer wrt  $K$  of  $F$  if  $\bar{y} \in \text{Min}(F(X))$ ;
- (ii) a weak minimizer wrt  $K$  of  $F$  if  $\bar{y} \in \text{WMin}(F(X))$ ;
- (iii) a Henig proper minimizer wrt  $\Theta$  of  $F$  if  $\bar{y} \in \text{He}(F(X))$ ;
- (iv) a super minimizer wrt  $K$  of  $F$  if  $\bar{y} \in \text{SE}(F(X))$ .

Next, we recall the concept of an approximate minimizer. Throughout this section let  $\epsilon > 0$  and  $k_0 \in K \setminus \{0\}$ .

**Definition 4.2.** We say that  $(\bar{x}, \bar{y}) \in \text{gr } F$  is an  $\epsilon k_0$ -minimizer wrt  $K$  of  $F$  over  $X$  iff  $\bar{y} \in \text{Min}(F(\bar{x}))$  and

$$y + \epsilon k_0 \not\leq_K \bar{y} \quad \text{for all } y \in F(X)$$

or equivalently,

$$(F(X) + \epsilon k_0 - \bar{y}) \cap (-K) = \emptyset.$$

Now, let us introduce the concepts of a Henig proper  $\epsilon k_0$ -minimizer wrt  $\Theta$  and of a super  $\epsilon k_0$ -minimizer wrt  $K$  for  $F$ .

**Definition 4.3.** We say that  $(\bar{x}, \bar{y}) \in \text{gr } F$  is

- (i) a Henig proper  $\epsilon k_0$ -minimizer wrt  $\Theta$  of  $F$  over  $X$  if  $\bar{y} \in \text{He}(F(\bar{x}))$  and there exists  $\eta \in ]0, \delta[$  such that

$$\text{cl cone}(F(X) + \epsilon k_0 - \bar{y}) \cap (-K_\eta) = \{0\};$$

- (ii) a super  $\epsilon k_0$ -minimizer wrt  $K$  of  $F$  over  $X$  if  $\bar{y} \in \text{SE}(F(\bar{x}))$  and there exists  $\rho > 0$  such that

$$\text{cl cone}(F(X) + \epsilon k_0 - \bar{y}) \cap (B - K) \subseteq \rho B.$$

One can check that Definition 4.3 reduces to Definition 4.1(iii)–(iv) when  $\epsilon = 0$ .

In [1], Amahroq, Penot and Syam introduced the notion of  $\epsilon k_0$ -blunt minimizers of a function. Motivated by their concept, we present the following definitions of a Henig proper  $\epsilon k_0$ -blunt minimizer and a super  $\epsilon k_0$ -blunt minimizer for the set-valued map  $F$ .

**Definition 4.4.** We say that  $(\bar{x}, \bar{y}) \in \text{gr } F$  is an  $\epsilon k_0$ -blunt minimizer wrt  $K$  (a Henig proper  $\epsilon k_0$ -blunt minimizer wrt  $\Theta$ , a super  $\epsilon k_0$ -blunt minimizer wrt  $K$ ) of  $F$  over  $X$  if it is a minimizer wrt  $K$  (respectively, a Henig proper minimizer wrt  $\Theta$ , a super minimizer wrt  $K$ ) of the perturbed set-valued map  $x \mapsto F(x) + \epsilon \|x - \bar{x}\| k_0$ .

Below is a variant of EVP established in [9] for an  $\epsilon k_0$ -minimizer of  $F$  that will be used later. Recall that  $F$  is *bounded from below* wrt  $K$  if there is  $v \in Y$  such that  $v \leq_K y$  for all  $y \in F(X)$  or equivalently, if  $F(X) \subset v + K$ .

**Theorem 4.1.** Assume that  $F$  is compact-valued, usc, and bounded from below wrt  $K$ . Suppose that  $\epsilon > 0$  and that  $k_0 \in K \setminus \{0\}$ . For any  $\tilde{y} \in F(X)$  there exists an  $\epsilon k_0$ -minimizer wrt  $K$ , say  $(x_\epsilon, y_\epsilon)$ , of  $F$  such that

- (a)  $y_\epsilon \leq_K \tilde{y}$ .
- (b)  $(x_\epsilon, y_\epsilon)$  is a strict minimizer wrt  $K$  of the perturbed set-valued map  $x \mapsto F(x) + \epsilon \|x - x_\epsilon\| k_0$  in the following sense

$$y + \epsilon \|x - x_\epsilon\| k_0 \not\leq_K y_\epsilon$$

for all  $(x, y) \in \text{gr } F$ ,  $(x, y) \neq (x_\epsilon, y_\epsilon)$ .

In other words,  $(x_\epsilon, y_\epsilon)$  in the above theorem is both an  $\epsilon k_0$ -minimizer and an  $\epsilon k_0$ -blunt minimizer of  $F$ . Our version of EVP for approximate Henig proper minimizers of  $F$  reads as follows.



**Theorem 4.2.** Assume that  $F$  is compact-valued, usc, and bounded from below wrt  $K$ . Suppose that  $\epsilon > 0$  and that  $k_0 \in K$  with  $\|k_0\| = 1$ . Then for any  $\eta \in ]0, \delta[$  and  $\tilde{y} \in F(X)$ , there exists a Henig proper  $\epsilon k_0$ -minimizer wrt  $\Theta$ , say  $(x_\epsilon, y_\epsilon)$ , of  $F$  such that

- (a)  $y_\epsilon \leq_{K_\eta} \tilde{y}$ , i.e.,  $\tilde{y} - y_\epsilon \in K_\eta$ .
- (b)  $(x_\epsilon, y_\epsilon)$  is a Henig proper minimizer wrt  $\Theta$  of the perturbed set-valued map  $x \mapsto F(x) + \epsilon \|x - x_\epsilon\| k_0$ .
- (c) There exist  $x_\epsilon^* \in X^*$  and  $y_\epsilon^* \in K^{+i}$  such that

$$\|y_\epsilon^*\| = 1, \quad \|x_\epsilon^*\| \leq (\delta/\eta)\epsilon$$

and

$$(x_\epsilon^*, -y_\epsilon^*) \in N((x_\epsilon, y_\epsilon); \text{gr } F)$$

or equivalently,

$$D^*F(x_\epsilon, y_\epsilon)(y_\epsilon^*) \cap (\delta/\eta)\epsilon B_{X^*} \neq \emptyset.$$

In other words,  $(x_\epsilon, y_\epsilon)$  in the above theorem is both a Henig proper  $\epsilon k_0$ -minimizer and a Henig proper  $\epsilon k_0$ -blunt minimizer of  $F$ . Note that we impose the requirement  $\|k_0\| = 1$  in the formulation of the theorem to get a “nicer” coderivative condition.

Let us prove an auxiliary result.

**Lemma 4.1.** Let be given  $k_0 \in K$  with  $\|k_0\| = 1$ . Then the following implication holds

$$y + \epsilon k_0 \not\leq_{K_\eta} y' \Rightarrow 0 < \Delta_{-K_\eta}(y - y') + \epsilon.$$

**Proof.** From  $k_0 \in K$  and  $\|k_0\| = 1$  we deduce that  $d_{-K_\eta}(k_0) \leq d_{-K}(k_0) \leq \|k_0\| = 1$ . Therefore,  $\Delta_{-K_\eta}(\epsilon k_0) = \epsilon d_{-K_\eta}(k_0) \leq \epsilon$ . Further, since  $K_\eta$  is closed and  $y - y' + \epsilon k_0 \notin -K_\eta$ , we have  $\Delta_{-K_\eta}(y - y' + \epsilon k_0) > 0$ . The triangle inequality (Proposition 3.4(iii)) yields

$$\begin{aligned} 0 &< \Delta_{-K_\eta}(y - y' + \epsilon k_0) \leq \Delta_{-K_\eta}(y - y') + \Delta_{-K_\eta}(\epsilon k_0) \\ &= \Delta_{-K_\eta}(y - y') + \epsilon d_{-K_\eta}(k_0) \leq \Delta_{-K_\eta}(y - y') + \epsilon, \end{aligned}$$

as it was to be shown.  $\square$

**Proof of Theorem 4.2.** It is obvious that if  $F$  is bounded from below wrt  $K$  then it is bounded from below wrt  $K_\eta$ . Applying Theorem 4.1 to  $F$  with  $K_\eta$  in the place of  $K$  we find  $(x_\epsilon, y_\epsilon)$ , an  $\epsilon k_0$ -minimizer wrt  $K_\eta$  of  $F$ , such that (a) holds and  $(x_\epsilon, y_\epsilon)$  is a strict minimizer wrt  $K_\eta$  of the perturbed set-valued map  $x \mapsto F(x) + \epsilon \|x - x_\epsilon\| k_0$ , i.e.,

$$y + \epsilon \|x - x_\epsilon\| k_0 \not\leq_{K_\eta} y_\epsilon \quad \text{for all } y \in F(X). \quad (8)$$

By Definition 4.2,  $y_\epsilon \in \text{Min}(F(x_\epsilon); K_\eta)$  and  $(F(X) + \epsilon k_0 - y_\epsilon) \cap (-K_\eta) = \emptyset$ . Then Proposition 3.3(i) yields that  $y_\epsilon \in \text{He}(F(x_\epsilon))$  and Proposition 3.1(i) yields that

$$\text{cl cone}(F(X) + \epsilon k_0 - y_\epsilon) \cap (-K_{\eta'}) = \{0\}$$

for any  $\eta' \in ]0, \eta[$ . Therefore,  $(x_\epsilon, y_\epsilon)$  is a Henig proper  $\epsilon k_0$ -minimizer wrt  $\Theta$  of  $F$ . Moreover, Proposition 3.3(i) again yields that  $(x_\epsilon, y_\epsilon)$  is a Henig proper minimizer wrt  $\Theta$  of the perturbed set-valued map. We have showed that (b) holds. Further, (8) and Lemma 4.1 give

$$\Delta_{-K_\eta}(y - y_\epsilon) + \epsilon \|x - x_\epsilon\| > 0 \quad \text{for all } (x, y) \in \text{gr } F, (x, y) \neq (x_\epsilon, y_\epsilon),$$

which means that  $(x_\epsilon, y_\epsilon)$  is the unique minimizer of the function  $g: X \times Y \rightarrow R \cup \{+\infty\}$  defined by

$$g(x, y) = \Delta_{-K_\eta}(y - y_\epsilon) + \epsilon \|x - x_\epsilon\| + \chi_{\text{gr } F}(x, y).$$

By the assumption, the map  $F$  has a closed graph and therefore, the function  $\chi_{\text{gr } F}$  is lsc. Then Proposition 2.1 implies that

$$0 \in \partial g(x_\epsilon, y_\epsilon)$$

and

$$\partial g(x_\epsilon, y_\epsilon) \subseteq \{0\} \times \partial \Delta_{-K_\eta}(0) + \epsilon B_{X^*} \times \{0\} + N((x_\epsilon, y_\epsilon); \text{gr } F).$$

Taking into account Proposition 3.5, we obtain

$$\partial g(x_\epsilon, y_\epsilon) \subseteq \{0\} \times (K^{+i} \cap (B_{Y^*} \setminus (\delta/\eta)B_{Y^*})) + \epsilon B_{X^*} \times \{0\} + N((x_\epsilon, y_\epsilon); \text{gr } F).$$

Hence, there exist  $\tilde{x}_\epsilon^* \in B_{X^*}$  and  $\tilde{y}_\epsilon^* \in K^{+i}$  such that  $\|\tilde{x}_\epsilon^*\| \leq \epsilon$ ,  $\|\tilde{y}_\epsilon^*\| \geq \eta/\delta$  and

$$(\tilde{x}_\epsilon^*, -\tilde{y}_\epsilon^*) \in N((x_\epsilon, y_\epsilon); \text{gr } F).$$

Denote  $x_\epsilon^* = \tilde{x}_\epsilon^*/\|\tilde{y}_\epsilon^*\|$ ,  $y_\epsilon^* = \tilde{y}_\epsilon^*/\|\tilde{y}_\epsilon^*\|$ . It is easy to check that  $\|x_\epsilon^*\| \leq (\delta/\eta)\epsilon$ ,  $y_\epsilon^* \in K^{+i}$ ,  $\|y_\epsilon^*\| = 1$  and

$$(x_\epsilon^*, -y_\epsilon^*) \in N((x_\epsilon, y_\epsilon); \text{gr } F)$$

or equivalently,

$$D^*F(x_\epsilon, y_\epsilon)(y_\epsilon^*) \cap (\delta/\eta)\epsilon B_{X^*} \neq \emptyset.$$

Thus, the assertion (c) is true.  $\square$

Our version of EVP for approximate super minimizers of  $F$  reads as follows.

**Theorem 4.3.** Assume that  $K$  has a bounded base and that  $F$  is compact-valued, usc, and bounded from below wrt  $K$ . Suppose that  $\epsilon > 0$  and that  $k_0 \in K$  with  $\|k_0\| = 1$ . Then for any  $\eta \in ]0, \delta[$  and  $\tilde{y} \in F(X)$ , there exists a super  $\epsilon k_0$ -minimizer wrt  $K$ , say  $(x_\epsilon, y_\epsilon)$ , of  $F$  such that

- (a)  $y_\epsilon \leq_{K_\eta} \tilde{y}$ .
- (b)  $(x_\epsilon, y_\epsilon)$  is a super minimizer wrt  $K$  of the perturbed set-valued map  $x \mapsto F(x) + \epsilon\|x - x_\epsilon\|k_0$ .
- (c) There exist  $x_\epsilon^* \in X^*$  and  $y_\epsilon^* \in \text{int } K^+$  such that

$$\|y_\epsilon^*\| = 1, \quad \|x_\epsilon^*\| \leq (\delta/\eta)\epsilon$$

and

$$(x_\epsilon^*, -y_\epsilon^*) \in N((x_\epsilon, y_\epsilon); \text{gr } F)$$

or equivalently,

$$D^*F(x_\epsilon, y_\epsilon)(y_\epsilon^*) \cap (\delta/\eta)\epsilon B_{X^*} \neq \emptyset.$$

In other words,  $(x_\epsilon, y_\epsilon)$  in the above theorem is both a super  $\epsilon k_0$ -minimizer and a super  $\epsilon k_0$ -blunt minimizer of  $F$ .

**Proof.** Apply the same arguments as in the proof of Theorem 4.2 and take into account the statements concerning the case  $\Theta$  is bounded in Propositions 3.2, 3.3 and 3.5.  $\square$

Let us return to the Banach spaces with the ordering cones being the nonnegative orthants considered in Example 3.1. Remark that Theorem 4.2 can be applied to set-valued maps taking values in  $R^n$ ,  $C_{[0,1]}$ ,  $L_{[0,1]}^p$  ( $1 \leq p < \infty$ ),  $m$ ,  $c$  and  $c_0$  and Theorem 4.3 can be applied to set-valued maps taking values in  $R^n$ ,  $L_{[0,1]}^1$  and  $l^1$ . Let us illustrate Theorems 4.2 and 4.3 by some examples.

**Example 4.1.** 1. Let  $X = c_0 \times m$ ,  $Y = c_0$ . For  $u = \{u_1, u_2, \dots\}$  and  $v = \{v_1, v_2, \dots\}$  in any of the spaces  $m$  or  $c_0$ ,  $uv$  means  $\{u_1v_1, u_2v_2, \dots\}$  and (supposing  $u_i \geq 0$  for all  $i$ )  $\sqrt{u}$  means  $\{\sqrt{u_1}, \sqrt{u_2}, \dots\}$ . Let  $K$  be the nonnegative orthant in  $c_0$  consisting of nonnegative sequences. We claim that this cone has an empty interior. Arguing by contradiction, suppose that there is  $k = \{k_1, k_2, \dots\} \in \text{int } K$  with  $\|k\| = 1$ . Note that we also have  $\sqrt{k} \in c_0$  and  $\|\sqrt{k}\| = 1$ . As  $k \in \text{int } K$ , there exists a scalar  $\zeta > 0$  such that  $k + \zeta B \subset K$ . However, we have  $k - \zeta\sqrt{k} \notin K$ . Indeed, if  $k - \zeta\sqrt{k} \in K$ , then we should have  $k_i - \zeta\sqrt{k_i} \geq 0$  for all  $i = 1, 2, \dots$ , which should yield  $\sqrt{k_i} \geq \zeta$  for all  $i = 1, 2, \dots$ . This is a contradiction to  $\sqrt{k} \in c_0$  and thus justifies  $\text{int } K = \emptyset$ . Further, given  $\xi = (\xi_1, \xi_2, \dots) \in c_0$  we define  $\varphi(\xi) = \xi_1 + \xi_2/2 + \dots + \xi_n/2^n + \dots$ . It is easy to check that  $\varphi \in K^{+i}$  and  $\Theta := \{k \in K : \varphi(k) = 1\}$  is a base of  $K$ .

Let  $\omega$  and  $\tau$  be arbitrary fixed vectors in  $K \setminus \{0\}$ . Put  $S = \{t\omega : t \in R_+\}$ . Then  $S$  is a compact set in  $c_0$ . We define a set-valued map  $F$  as follows

$$F(x_1, x_2) = f(x_1, x_2) + S \quad \text{for all } (x_1, x_2) \in c_0 \times m,$$

where  $f(x_1, x_2) := (x_1)^2 + (x_1x_2 - \tau)^2$ , i.e.,  $y \in F(x_1, x_2)$  means the existence of  $v \in S$  such that  $y = (x_1)^2 + (x_1x_2 - \tau)^2 + v$ . One can check that the map  $F$  is compact-valued, usc, and bounded from below ( $F(X) \subset K$ ). Therefore, Theorem 4.2 is applicable to this map. Observe by passing that  $F$  has no minimizer. Indeed, fix  $\bar{y} \in F(X)$ , i.e.,  $\bar{y} = f(\bar{x}_1, \bar{x}_2) + \bar{v}$  for some  $(\bar{x}_1, \bar{x}_2) \in c_0 \times m$  and  $\bar{v} \in S$ . Let  $\tilde{x}_1 = \tau/4 \in c_0$ ,  $\tilde{x}_2 = \{1, 1, 1, \dots\} \in m$ . Let  $\tilde{y} \in c_0$  be a vector chosen as follows

$$\tilde{y} = \begin{cases} f(\bar{x}_1/2, 2\bar{x}_2) + \bar{v} & \text{if } \bar{x}_1 \neq 0, \\ f(\bar{x}_1, \bar{x}_2) + \bar{v}/2 & \text{if } \bar{x}_1 = 0, \bar{v} \neq 0, \\ f(\bar{x}_1, \bar{x}_2) & \text{if } \bar{x}_1 = 0, \bar{v} = 0. \end{cases}$$

One has  $\tilde{y} \in F(X)$ ,  $\tilde{y} \neq \bar{y}$  and  $\tilde{y} \leq \bar{y}$ . This means that  $\bar{y}$  cannot be an efficient point of  $F(X)$ . Thus,  $F$  has no minimizer and no Henig proper minimizer as well.

We refer the interested readers to [21] for an example of a set-valued map with values in  $C_{[0,1]}$  which satisfies all conditions of Theorem 4.2 while it does not possess any Henig proper minimizer.

2. Similarly, one can provide an example of a set-valued map with values in  $l^1$  or  $L^1_{[0,1]}$  (the nonnegative orthants in these spaces have a bounded base) which does not possess any super minimizer but satisfies all conditions of Theorem 4.3 and therefore has approximate super minimizers. We provide an example for the case  $Y = L^1_{[0,1]}$ . Let  $X = C_{[0,1]} \times C_{[0,1]}$ ,  $Y = L^1_{[0,1]}$ , and  $K$  be the nonnegative orthant in  $L^1_{[0,1]}$  consisting of a.e. nonnegative functions. One can check that  $K$  has an empty interior and that  $K$  has a bounded base  $\Theta := \{x \in K, \int_0^1 x(t) dt = 1\}$ . Let  $S \subset K$  be a nonempty compact set defined by  $S = \{v \in L^1_{[0,1]} : v(t) = \int_0^t u(s) ds, u \in C_{[0,1]}, 0 \leq u(t) \leq 1 \text{ for all } t \in [0, 1]\}$ . We define  $F$  as follows

$$F(x_1, x_2) = (x_1)^2 + (x_1 x_2 - 1)^2 + S \quad \text{for all } (x_1, x_2) \in C_{[0,1]} \times C_{[0,1]},$$

i.e.,  $y \in F(x_1, x_2)$  means the existence of  $v \in S$  such that  $y(t) = (x_1(t))^2 + (x_1(t)x_2(t) - 1)^2 + v(t)$  for all  $t \in [0, 1]$ . The map  $F$  is compact-valued, usc, and bounded from below ( $0 \leq_K y$  for all  $y \in F(X)$ ). Therefore, Theorem 4.3 is applicable to this map. By an argument similar to that used in Example 3.1 of [21] one can prove that  $F$  has no minimizer and therefore, it has no super minimizer.

Observe that  $c_0$ ,  $l^1$  and  $L^1_{[0,1]}$  are not Asplund spaces and the nonnegative orthants in these spaces have an empty interior. Hence, the existing set-valued versions of EVP obtained either under the assumption that the ordering cone has a nonempty interior or in Asplund space setting cannot be applied to the maps in Example 4.1.

**Remark 4.1.** We note that the concepts of the Henig proper efficient points and super efficient points in a locally convex space have been defined and investigated in [52]. A question arises: how to extend the statements (i)–(ii) of Theorems 4.2 and 4.3 to the case when  $F$  takes values in such a space.

## 5. Sufficient conditions for the existence of Henig proper minimizers and super minimizers

In this section we apply Theorems 4.2 and 4.3 to obtain the existence of Henig proper minimizers and super minimizers for a set-valued map satisfying some Palais–Smale type conditions.

Motivated by the modified versions of the Palais–Smale condition which are defined for a single-valued function and expressed in terms of its subdifferential [2] or for a set-valued map and expressed in terms of its subdifferential [4,5] or its coderivatives [19,21], we introduce the following *Palais–Smale type conditions*.

(PS1) Every sequence  $\{x_n\}$  in  $X$  such that

$$\begin{aligned} &\text{there are } y_n \in F(x_n), y_n^* \in K^{+i} \text{ with } \|y_n^*\| = 1, \\ &x_n^* \in D^*F(x_n, y_n)(y_n^*) \text{ with } \|x_n^*\| \rightarrow 0 \end{aligned}$$

contains a convergent subsequence, provided that  $\{y_n\}$  is norm bounded.

(PS2) Every sequence  $\{x_n\}$  in  $X$  such that

$$\begin{aligned} &\text{there are } y_n \in F(x_n), y_n^* \in \text{int } K^+ \text{ with } \|y_n^*\| = 1, \\ &x_n^* \in D^*F(x_n, y_n)(y_n^*) \text{ with } \|x_n^*\| \rightarrow 0 \end{aligned}$$

contains a convergent subsequence, provided that  $\{y_n\}$  is norm bounded.

We note that the above conditions agree with the classical Palais–Smale condition when  $F$  is a (single-valued) Fréchet function from  $X$  to  $\mathbb{R}$ .

Our sufficient condition for  $F$  to have a Henig proper minimizer reads as follows.

**Theorem 5.1.** Assume that  $K$  is normal and that  $F$  is compact-valued, usc, bounded from below wrt  $K$ , and satisfies (PS1). Then  $F$  has a Henig proper minimizer wrt  $\Theta$ .

Let us prove an auxiliary result.

**Lemma 5.1.** Assume that  $K$  is normal with the constant  $N$ . Let  $\eta \in ]0, \delta[$  be a scalar such that  $\eta \leq \delta/(2N)$ . Let  $u, v \in Y$  such that

$$0 \leq_K u \leq_{K_\eta} v.$$

Then one has

$$\|u\| \leq 2N(\|v\| + 1).$$

**Proof.** As  $v - u \in K_\eta$ , there exist a scalar  $\lambda > 0$ , vectors  $\theta \in \Theta$ ,  $b \in B$  and  $z \in B$  such that  $u = v - \lambda(\theta + \eta b) - z$ . So we have

$$0 \leq_K u = v - \lambda(\theta + \eta b) - z \leq_K v - \lambda\eta b - z.$$

Put  $\omega = v - \lambda\eta b - z$ . Then we have

$$\|\omega\| \leq \|v\| + \lambda\eta\|b\| + \|z\| \leq \|v\| + \lambda\eta + 1$$

and hence,

$$N\|\omega\| \leq N(\|v\| + 1) + N\lambda\eta \leq N(\|v\| + 1) + \lambda\delta/2.$$

Since  $\delta \leq \|\theta\|$ ,  $0 \leq_K \lambda\theta \leq_K \omega$  and the cone  $K$  is normal, we obtain

$$\lambda\delta \leq \|\lambda\theta\| \leq N\|\omega\| \leq N(\|v\| + 1) + \lambda\delta/2,$$

which yields  $\lambda\delta/2 \leq N(\|v\| + 1)$ . It follows that  $N\|\omega\| \leq 2N(\|v\| + 1)$ . Finally, as  $0 \leq_K u \leq_K \omega$  and the cone  $K$  is normal, we get

$$\|u\| \leq N\|\omega\| \leq 2N(\|v\| + 1)$$

as it was to be shown.  $\square$

**Proof of Theorem 5.1.** We adapt the technique used in the proof of Theorem 4.1 in [21]. Suppose that  $k_0 \in K \setminus \{0\}$  with  $\|k_0\| = 1$  and that  $\tilde{y} \in F(X)$ . Put  $\epsilon_n = 2^{-n}$  for all  $n = 1, 2, \dots$ . Let  $\eta$  be a scalar satisfying  $\eta = \min\{\delta/2, \delta/(2N)\}$ . By Theorem 4.2, there exist  $(x_n, y_n) \in \text{gr } F$  and  $(x_n^*, -y_n^*) \in N((x_n, y_n); \text{gr } F)$  such that  $y_n \leq_{K_\eta} \tilde{y}$ ,  $y_n^* \in K^{+i}$ ,  $\|y_n^*\| = 1$  and

$$\|x_n^*\| \leq (\delta/\eta)\epsilon_n. \quad (9)$$

Suppose that  $w \in Y$  be the vector satisfying  $w \leq_K y$  for all  $y \in F(X)$ . Then  $w \leq_K y_n \leq_{K_\eta} \tilde{y}$  for all  $n = 1, 2, \dots$  or  $0 \leq_K y_n - w \leq_{K_\eta} \tilde{y} - w$ . By Lemma 5.1, the sequence  $\{y_n - w\}$  is bounded and so is the sequence  $\{y_n\}$ . Further, (9) yields that  $\|x_n^*\| \rightarrow 0$ . By (PS1), there exist  $\bar{x} \in X$  and a subsequence  $\{x_{n_i}\}_{i=1}^\infty$  converging to  $\bar{x}$ . Since  $F$  is usc, for each  $i = 1, 2, \dots$  there exists a positive integer  $m_i$  such that

$$F(x_{n_j}) \subset F(\bar{x}) + 2^{-i}B_Y \quad \text{for all } n_j > m_i.$$

Passing to a subsequence if necessary, we can assume that

$$F(x_{n_i}) \subset F(\bar{x}) + 2^{-i}B_Y \quad \text{for all } i = 1, 2, \dots$$

Hence, one can find  $\bar{y}_{n_i} \in F(\bar{x})$  for each  $i = 1, 2, \dots$  such that

$$\|\bar{y}_{n_i} - y_{n_i}\| \leq 2^{-i} \quad \text{for all } i = 1, 2, \dots \quad (10)$$

As  $(x_n, y_n)$  are  $\epsilon_n k_0$ -minimizers wrt  $K_\eta$  of  $F$ , one has

$$y + \epsilon_n k_0 \not\leq_{K_\eta} y_n \quad \text{for all } y \in F(X).$$

Lemma 4.1 then implies that for all  $n = 1, 2, \dots$

$$\Delta_{-K_\eta}(y - y_n) + \epsilon_n > 0 \quad \text{for all } y \in F(X)$$

or

$$\Delta_{-K_\eta}(y - y_n) + 2^{-n} > 0 \quad \text{for all } y \in F(X). \quad (11)$$

We claim that for all  $i = 1, 2, \dots$  one has

$$\Delta_{-K_\eta}(y - \bar{y}_{n_i}) + 2^{-n_i} + 2^{-i} > 0 \quad \text{for all } y \in F(X). \quad (12)$$

Indeed, fix  $i$  and  $y \in F(X)$ . If  $\bar{y}_{n_i} - y_{n_i} \in -K_\eta$  then the triangle inequality and (11) yield

$$\begin{aligned}\Delta_{-K_\eta}(y - \bar{y}_{n_i}) &\geq \Delta_{-K_\eta}(y - y_{n_i}) - \Delta_{-K_\eta}(\bar{y}_{n_i} - y_{n_i}) \\ &\geq -2^{-n_i} + d_{Y \setminus (-K_\eta)}(\bar{y}_{n_i} - y_{n_i}) \geq -2^{-n_i} > -2^{-n_i} - 2^{-i}\end{aligned}$$

and if  $\bar{y}_{n_i} - y_{n_i} \notin -K_\eta$  then the triangle inequality and (10) yield

$$\begin{aligned}\Delta_{-K_\eta}(y - \bar{y}_{n_i}) &\geq \Delta_{-K_\eta}(y - y_{n_i}) - \Delta_{-K_\eta}(\bar{y}_{n_i} - y_{n_i}) \\ &> -2^{-n_i} - d_{-K_\eta}(\bar{y}_{n_i} - y_{n_i}) \geq -2^{-n_i} - d(0, \bar{y}_{n_i} - y_{n_i}) \\ &\geq -2^{-n_i} - 2^{-i}.\end{aligned}$$

Thus, (12) holds.

Further, since  $F(\bar{x})$  is compact and  $\bar{y}_{n_i} \in F(\bar{x})$  for all  $i = 1, 2, \dots$ , without loss of generality we can assume that  $\bar{y}_{n_i}$  converges to some  $\bar{y} \in F(\bar{x})$ . Letting  $i \rightarrow \infty$  in (12) we get

$$\Delta_{-K_\eta}(y - \bar{y}) \geq 0 \quad \text{for all } y \in F(X).$$

By Proposition 3.5(i),  $(\bar{x}, \bar{y})$  is a Henig proper minimizer wrt  $\Theta$  of  $F$ .  $\square$

Our sufficient condition for  $F$  to have a super minimizer reads as follows.

**Theorem 5.2.** Assume that  $K$  has a bounded base and that  $F$  is compact-valued, usc, bounded from below wrt  $K$ , and satisfies (PS2). Then  $F$  has a super minimizer wrt  $K$ .

**Proof.** Recall that a cone having a bounded base is normal. To complete the proof it suffices to take into account Theorem 4.3, Lemma 5.1, Proposition 3.5(ii) and apply the same arguments as in the proof of Theorem 5.1.  $\square$

**Remark 5.1.** In [4,5] Bao and Mordukhovich deduced from their subdifferential EVP the existence of relative Pareto minimizers under very weak conditions on the considered set-valued map  $F$  and in Asplund space settings. In contrast to Bao and Mordukhovich's work, our conditions in Theorems 5.1 and 5.2 on  $F$  may be stronger but these theorems can be applied to Banach spaces which are not Asplund. Namely, Theorem 5.1 can be applied to the case when  $F$  takes values in  $R^n$ ,  $C_{[0,1]}$ ,  $L^p_{[0,1]}$ ,  $l^p$  ( $1 \leq p < \infty$ ),  $m$ ,  $c$  and  $c_0$  and Theorem 5.2 can be applied to the case when  $F$  takes values in  $R^n$ ,  $L^1_{[0,1]}$  and  $l^1$ .

Finally, we derive from Theorems 5.1 and 5.2 some optimality conditions for the case when  $F$  is single-valued and is Fréchet differentiable on  $X$ . Denote the Fréchet derivative of  $F$  at  $x$  by  $F'(x)$ . It is known that

$$D^*F(x, F(x))(y^*) = \{F'(x)^*(y^*)\} \quad \text{for any } y^* \in Y^*,$$

where  $F'(x)^*$  is the adjoint operator of  $F'(x)$ .

As immediate corollaries of Theorems 5.1 and 5.2, we have:

**Corollary 5.1.** Assume that  $K$  is normal. Assume further that  $F$  is Fréchet differentiable on  $X$ , bounded from below wrt  $K$ , and satisfies the following Palais–Smale type condition:

Every sequence  $\{x_n\}$  in  $X$  such that

$$\text{there are } y_n^* \in K^{+i} \text{ with } \|y_n^*\| = 1 \text{ and } \|F'(x_n)^*(y_n^*)\| \rightarrow 0$$

contains a convergent subsequence, provided that  $\{F(x_n)\}$  is norm bounded.

Then  $F$  has a Henig proper minimizer wrt  $\Theta$ , i.e., there exists  $\bar{x} \in X$  such that  $F(\bar{x}) \in \text{He}(F(X))$ .

**Corollary 5.2.** Assume that  $K$  has a bounded base. Assume further that  $F$  is Fréchet differentiable on  $X$ , bounded from below wrt  $K$ , and satisfies the following Palais–Smale type condition:

Every sequence  $\{x_n\}$  in  $X$  such that

$$\text{there are } y_n^* \in \text{int } K^+ \text{ with } \|y_n^*\| = 1 \text{ and } \|F'(x_n)^*(y_n^*)\| \rightarrow 0$$

contains a convergent subsequence, provided that  $\{F(x_n)\}$  is norm bounded.

Then  $F$  has a super minimizer wrt  $K$ , i.e., there exists  $\bar{x} \in X$  such that  $F(\bar{x}) \in \text{SE}(F(X))$ .

Let us consider the case  $Y = R^m$  and  $K = R_+^m := \{(x_1, \dots, x_m) : x_i \geq 0, i = 1, 2, \dots, m\}$ . Since  $R_+^m$  has a bounded base  $\Theta := \{(x_1, \dots, x_m) : x_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m x_i = 1\}$ , the Henig proper efficiency coincides with the super efficiency. Moreover, we have

$$K^{+i} = \text{int } K^+ = R_{++}^m,$$

where  $R_{++}^m := \{(x_1, \dots, x_m) : x_i > 0, i = 1, 2, \dots, m\}$ . As an immediate consequence of Theorems 5.1 and 5.2 we have:

**Corollary 5.3.** Assume that  $F : X \rightarrow R^m$  is Fréchet differentiable on  $X$ , bounded from below wrt  $R_+^m$ , and satisfies the following Palais–Smale type condition:

Every sequence  $\{x_n\}$  in  $X$  such that

$$\text{there are } y_n^* \in R_{++}^m \text{ with } \|y_n^*\| = 1 \text{ and } \|F'(x_n)^*(y_n^*)\| \rightarrow 0$$

contains a convergent subsequence, provided that  $\{F(x_n)\}$  is norm bounded.

Then  $F$  has a Henig proper minimizer wrt  $\Theta$  and a super minimizer wrt  $R_+^m$ , i.e., there exists  $\bar{x} \in X$  such that  $F(\bar{x}) \in \text{He}(F(X)) = \text{SE}(F(X))$ .

## References

- [1] T. Amahroq, J.-P. Penot, A. Syam, On the subdifferentiability of the difference of two functions and local minimization, *Set-Valued Anal.* 16 (4) (2008) 413–427.
- [2] D. Aussel, J.-N. Corvellec, M. Lassonde, Mean value property and subdifferential criteria for lower semicontinuous function, *Trans. Amer. Math. Soc.* 347 (1995) 4147–4161.
- [3] I.A. Bakhtin, Cones in Banach Spaces, Part 1, Voronezh, 1975 (in Russian).
- [4] T.Q. Bao, B.S. Mordukhovich, Relative Pareto minimizers to multiobjective Problems: Existence and optimality conditions, *Math. Progr.* 122 (2) (2010) 301–347.
- [5] T.Q. Bao, B.S. Mordukhovich, Variational principles for set-valued mappings with applications to multiobjective optimization, *Control Cybernet.* 36 (2007) 531–562.
- [6] J.M. Borwein, D. Zhuang, Super efficiency in vector optimization, *Trans. Amer. Math. Soc.* 338 (1993) 105–122.
- [7] G.Y. Chen, X.X. Huang, A unified approach to the existing three types of variational principles for vector valued functions, *Math. Methods Oper. Res.* 48 (1998) 349–357.
- [8] G.Y. Chen, X.X. Huang, X. Yang, Vector optimization, in: *Set-Valued and Variational Analysis*, in: *Lecture Notes in Econom. and Math. Systems*, vol. 541, Springer-Verlag, Berlin, 2005.
- [9] G.Y. Chen, X.X. Huang, S.H. Hou, A general approximate variational principle for set-valued maps, *J. Optim. Theory Appl.* 106 (2000) 151–164.
- [10] F. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, 1983.
- [11] D. Dentcheva, S. Helbig, On variational principles, level sets, well-posedness, and  $\epsilon$ -solutions in vector optimization, *J. Optim. Theory Appl.* 89 (1996) 325–349.
- [12] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* 47 (1974) 324–353.
- [13] C. Finet, Variational principles in partially ordered Banach spaces, *J. Nonlinear Convex Anal.* 2 (2001) 167–174.
- [14] C. Finet, L. Quarta, C. Troestler, Vector-valued variational principles, *Nonlinear Anal.* 52 (1) (2003) 197–218.
- [15] V.V. Gorokhovich, *Convex and Nonsmooth Problems of Vector Optimization*, Nauka i Tekhnika, Minsk, Belarus, 1990 (in Russian).
- [16] A. Göpfert, H. Riahi, C. Tammer, C. Zalinescu, *Variational Methods in Partially Ordered Spaces*, Springer-Verlag, New York, 2003.
- [17] A. Göpfert, C. Tammer, C. Zalinescu, On the vectorial Ekeland variational principle and minimal points in product spaces, *Nonlinear Anal.* 39 (2000) 909–922.
- [18] C. Gutierrez, B. Jimenez, V. Novo, A set-valued Ekeland variational principle, *SIAM J. Control Optim.* 47 (2008) 883–903.
- [19] T.X.D. Ha, The Ekeland variational principle for set-valued maps involving coderivatives, *J. Math. Anal. Appl.* 286 (2003) 509–523.
- [20] T.X.D. Ha, Some variants of the Ekeland variational principle for a set-valued map, *J. Optim. Theory Appl.* 124 (2005) 187–206.
- [21] T.X.D. Ha, Variants of the Ekeland variational principle for a set-valued map involving the Clarke normal cone, *J. Math. Anal. Appl.* 316 (2006) 346–356.
- [22] T.X.D. Ha, Optimality conditions for several types of efficient solutions of set-valued optimization problems, in: P. Pardalos, Th.M. Rassias, A.A. Khan (Eds.), *Nonlinear Analysis and Variational Problems*, Springer, 2009, pp. 305–324.
- [23] A.H. Hamel, Phelps' lemma, Danes' drop theorem and Ekeland's principle in locally convex topological vector spaces, *Proc. Amer. Math. Soc.* 10 (2003) 3025–3038.
- [24] A.H. Hamel, Equivalents to Ekeland's variational principle in uniform spaces, *Nonlinear Anal.* 62 (2005) 913–924.
- [25] A.H. Hamel, A. Löhne, Minimal element theorems and Ekeland's principle with set relations, *J. Nonlinear Convex Anal.* 7 (2006) 19–37.
- [26] M.I. Henig, Proper efficiency with respect to the cones, *J. Optim. Theory Appl.* 36 (1982) 387–407.
- [27] J.B. Hiriart-Urruty, New concepts in nondifferentiable programming, *Bull. Soc. Math. France* 60 (1979) 57–85.
- [28] X.X. Huang, New stability results for Ekeland's variational principle for vector-valued and set-valued maps, *J. Math. Anal. Appl.* 262 (2001) 12–23.
- [29] D.H. Hyers, G. Isac, T.M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific Publishing, River Edge, NJ, 1997.
- [30] A.D. Ioffe, Approximate subdifferentials and applications 3: Metric theory, *Mathematica* 36 (1989) 1–38.
- [31] A.D. Ioffe, J.-P. Penot, Subdifferentials of performance functions and calculus of coderivatives of set-valued mappings, in: *Well-Posedness and Stability of Variational Problems*, *Serdica Math. J.* 22 (3) (1996) 257–282.
- [32] G. Isac, The Ekeland principle and the Pareto  $\epsilon$ -efficiency, in: *Multi-Objective Programming and Goal Programming: Theories and Applications*, in: *Lecture Notes in Econom. and Math. Systems*, vol. 432, Springer-Verlag, Berlin, 1996, pp. 148–163.
- [33] G. Isac, C. Tammer, Nuclear and full nuclear cones in product spaces: Pareto efficiency and an Ekeland type variational principle, *Positivity* 9 (2005) 511–539.
- [34] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer-Verlag, Berlin, 2004.
- [35] P.Q. Khanh, On Caristi–Kirk's theorem and Ekeland's variational principle for Pareto extrema, *Bull. Pol. Acad. Sci. Math.* 37 (1989) 1–6.
- [36] M.A. Krasnoselski, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, Holland, 1964.

- [37] A.Y. Kruger, B.S. Mordukhovich, Extremal points and the Euler equations in nonsmooth optimization, *Dokl. Akad. Nauk BSSR* 24 (1980) 684–687 (in Russian).
- [38] P. Loridan,  $\epsilon$ -Solutions in vector minimization problems, *J. Optim. Theory Appl.* 43 (1984) 265–276.
- [39] D.T. Luc, *Theory of Vector Optimization*, Springer-Verlag, Berlin, Germany, 1989.
- [40] E.K. Makarov, N.N. Rachkovski, Unified representation of proper efficiency by means of dilating cones, *J. Optim. Theory Appl.* 101 (1999) 141–165.
- [41] B.S. Mordukhovich, Maximum principle in problems of time optimal control with nonsmooth constraints, *J. Appl. Math. Mech.* 40 (1976) 960–969.
- [42] B.S. Mordukhovich, Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems, *Soviet Math. Dokl.* 22 (1980) 526–530.
- [43] B.S. Mordukhovich, Generalized differential calculus for nonsmooth and set-valued mappings, *J. Math. Anal. Appl.* 183 (1994) 250–288.
- [44] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*, Grundlehren Ser. (Fundamental Principles of Mathematical Sciences), vol. 330, Springer, Berlin, 2006.
- [45] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, II: Applications*, Grundlehren Ser. (Fundamental Principles of Mathematical Sciences), vol. 331, Springer, Berlin, 2006.
- [46] B.S. Mordukhovich, B. Wang, Necessary suboptimality and optimality conditions via variational principles, *SIAM J. Control Optim.* 41 (2002) 623–640.
- [47] A.B. Németh, A nonconvex vector minimization problem, *Nonlinear Anal.* 10 (1986) 669–678.
- [48] R.T. Rockafellar, *The Theory of Subgradients and Its Applications to Problems of Optimization: Convex and Nonconvex Functions*, Hedelman Verlag, Berlin, 1981.
- [49] C. Tammer, A generalization of Ekeland's variational principle, *Optimization* 25 (1992) 129–141.
- [50] A. Zaffaroni, Degrees of efficiency and degrees of minimality, *SIAM J. Control Optim.* 42 (2003) 1071–1086.
- [51] X.Y. Zheng, Scalarization of Henig proper efficient points in a normed space, *J. Optim. Theory Appl.* 105 (2000) 233–247.
- [52] X.Y. Zheng, Proper efficiency in local convex topological vector spaces, *J. Optim. Theory Appl.* 94 (1997) 469–486.
- [53] J. Zhu, C.K. Zhong, Y.J. Cho, Generalized variational principle and vector optimization, *J. Optim. Theory Appl.* 106 (2000) 201–217.
- [54] D. Zhuang, Density results for proper efficiency, *SIAM J. Control Optim.* 32 (1994) 51–58.